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# CONVERGENCE OF INTERPOLATING CARDINAL SPLINES: POWER GROWTH

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ABSTRACT

Let $f(x)$ be the restriction to the real axis of an entire function of exponential type $\tau < \pi$ and of power growth on the axis. Then the $n$th order cardinal spline, $\mathscr{L}_n f(x)$, interpolating $f(x)$ at the integers converges uniformly on compacta to $f(x)$. This is also true of the respective derivatives. An example shows that exponential type $\pi$ is not necessarily permitted. The proof utilizes distribution theory and estimates on the derivatives of the Fourier transform of the fundamental spline $L_n(x)$.

For a natural number $n$, the space $\mathscr{S}_n \cap F_s = \{S(x)\}$ of cardinal splines with power growth and of degree $n-1$ is taken to consist of those functions satisfying:

i) $S \in C^{n-2}(-\infty, +\infty)$;

ii) $|S(x)| = \mathcal{O}(|x|^s)$ for some $s \geqq 0$;

iii) $S(x)$ reduces to to a polynomial of degree at most $n-1$ on each of the intervals $[\nu + (n/2), v + (n/2) + 1]$, $\nu \in \mathbf{Z}$, i.e. $S(x)$ has knots at the integers or half integers if $n$ is respectively even or odd.

For a sequence $y = \{y_\nu\}_{\nu=-\infty}^{+\infty} \in Y_s$, $Y_s = \{y : y_\nu = \mathcal{O}(|\nu|^s) \mid \nu| \to +\infty\}$, there is a unique element $\mathscr{L}_n y \in \mathscr{S}_n \cap F_s$ interpolating the given data at the integers, i.e. $\mathscr{L}_n y(\nu) = y_\nu$, $\nu \in \mathbf{Z}$. The present paper deals with the following questions: Suppose that the data $y = \{y_\nu\}_{\nu=-\infty}^{+\infty}$ arises from some "suitable" function $f$ by $y_\nu = f(\nu)$, $\nu \in \mathbf{Z}$. When does $\mathscr{L}_n f(x)$ converge to $f(x)$ and in what sense is the convergence? What are the "suitable" functions?

Questions of this type have been studied by I. J. Schoenberg [7], [8], [9], Richards and Schoenberg [5], and Marsden, Richards and Riemenschneider [3]. In the monograph [9], Schoenberg raises the question as to the existence of a

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comprehensive theory that would cover the various known cases of convergence. In this report, some new results are given which are based on the work of R. R. Goldberg [1] and which is believed to lead towards a comprehensive theory with the aid of distribution theory.

In Section 2, the uniform convergence on bounded sets of $\mathcal{L}_n f(x)$ to $f(x)$ is shown in the case when $f(x)$ is the restriction to $\mathbf{R}$ of an entire function of exponential type $\tau < \pi$ with power growth on the real axis. In the final section, an example is given to show that the above theorem cannot be strengthened to allow exponential type $\pi$.

## 1. Preliminaries

The fundamental cardinal spline, $L_n(x)$, is the unique element of $\mathcal{S}_n$ which interpolates the data $y_\nu = 0$, $\nu \neq 0$, $y_0 = 1$. The unique element $\mathcal{L}_n y(x) \in \mathcal{S}_n \cap F_s$ interpolating $y \in Y_s$ admits the representation

$$\mathcal{L}_n y(x) = \sum_{\nu=-\infty}^{+\infty} y_\nu L_n(x-\nu), \qquad x \in \mathbf{R} \tag{1.1}$$

(Schoenberg [8]).

An important role is played by the Fourier transform representation of $L_n(x)$,

$$L_n(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\psi_n/\phi_n(u)]e^{iux}du \tag{1.2}$$

where $\psi_n(u) = [(2/u)\sin u/2]^n$ and $\phi_n(u) = \sum_{j=-\infty}^{+\infty} \psi_n(u + 2\pi j)$ (see Schoenberg [6], [7], and [8]). Its pertinence to convergence results was demonstrated in [3].

In [3], it was shown that $|(\psi_n/\phi_n)(u)| \leqq \min[1, (\pi/u)^n]$ and that $(\psi_n/\phi_n)(u)$ converges uniformly to $\chi_{[-\pi,\pi]}(u)$ outside arbitrarily small intervals about $\pm\pi$. The object of this section is to extend these results to the derivatives of $\psi_n/\phi_n$.

THEOREM 1.1. *Let s be a natural number* $\geqq 1$. *Then*

i) *for* $2\pi k - \pi \leqq u \leqq 2\pi k + \pi$, $k \in \mathbf{Z}$, $k \neq 0, \pm 1$,

$$\left|\left(\frac{\psi_n}{\phi_n}\right)^{(s)}(u)\right| = O\left(n^s\left(\frac{3}{2}\right)^{-n}|k|^{-n}\right) \tag{1.3}$$

*and*

ii) *for* $\varepsilon_{n,s} = \pi(s+1)\log n/n$, *and* $0 \leqq |u| \leqq \pi - \varepsilon_{n,s}$ *or* $\pi + \varepsilon_{n,s} \leqq |u| \leqq 3\pi$,

$$\left|\left(\frac{\psi_n}{\phi_n}\right)^{(s)}(u)\right| = O(1/n). \tag{1.4}$$

PROOF. It will be convenient to introduce the function

$$f(u;n)=\sum_{j=-\infty}^{+\infty}(-1)^{jn}[u-2\pi j]^{-n} \tag{1.5}$$

and to observe that for any integer $k$

$$f(u;n)=(-1)^{kn}\left\{(u-2\pi k)^{-n}+\sum_{j=1}^{\infty}C_{j,n}(u-2\pi k)\right\} \tag{1.6}$$

where $C_{j,n}(u)\geqq 0$ for $0<u\leqq\pi$ and $C_{j,n}(u)\leqq 0$ for $-\pi\leqq u<0$ (see [3, prop. 1.2]). Finally,

$$[u^n f(u;n)]^{-1}=(\psi_n/\phi_n)(u),\qquad u\neq 2\pi k. \tag{1.7}$$

Let $D$ denote the differentiation operator, and let the letter $A$ denote an absolute constant which is independent of $u$ and $n$ (but not of $s$) and which may change from line to line. Further, all inequalities are assumed to hold for $n$ sufficiently large.

For $2\pi k-\pi\leqq u\leqq 2\pi k+\pi$, $u\neq 2\pi k$, $k\neq 0$, (1.6) and (1.7) can be used to obtain

$$|D(\psi_n/\phi_n)(u)|\leqq A\,|k|\,n[\,|u^{n+1}f(u;n)(u-2\pi k)|\,]^{-1}.$$

By the periodicity of $|f(u;n)|$ and the fact that $|u^n f(u;n)|^{-1}\leqq 1$, it follows that

$$|D(\psi_n/\phi_n)(u)|\leqq A\,|k|\,n\,|u-2\pi k|^{n-1}|u|^{-n-1}, \tag{1.8}$$

and, consequently,

$$\begin{aligned}|D(\psi_n/\phi_n)(u)|&\leqq A\,|k|\,n\pi^{n-1}[\pi\min\{|2k-1|,|2k+1|\}]^{-n-1}\\&\leqq An\left(\frac{3}{2}\right)^{-n}|k|^{-n}\end{aligned}$$

for $2\pi k-\pi\leqq u\leqq 2\pi k+\pi$ and $k\neq 0,\pm 1$.

For $k=\pm 1$, and $\pi+\varepsilon_{n,1}\leqq|u|\leqq 3\pi$, equation (1.8) gives

$$|D(\psi_n/\phi_n)(u)|\leqq An\left(1-\frac{2\log n}{n}\right)^{n-1}\leqq A/n.$$

Finally, for $k=0$, and $0\leqq|u|\leqq\pi-\varepsilon_{n,1}$, notice that $\Sigma_{j\neq 0}(-1)^{jn}j[u-2\pi j]^{-n-1}=\mathcal{O}(\pi^{-n})$. Hence, we can obtain

$$\begin{aligned}|D(\psi_n/\phi_n)(u)|&\leqq An\pi^{-n}|u|^{n-1}\\&\leqq An\left(1-\frac{2\log n}{n}\right)^{n-1}\leqq A/n.\end{aligned} \tag{1.9}$$

The argument is carried over to arbitrary $s$ by induction on equations of the form (1.8) and (1.9). First assume that for $2\pi k - \pi \leqq u \leqq 2\pi k + \pi$, $u \neq 2\pi k$, $k \neq 0$, and any $m$, $1 \leqq m < s$, there holds

$$(1.10) \qquad |D^m(\psi_n/\phi_n)(u)| \leqq A\,|k|\,n^m\,|u - 2\pi k|^{n-m}\,|u|^{-n-1}.$$

From the nature of the function $f(u;n)$, Leibnitz's rule yields

$$(1.11) \qquad |D^l[u^n f(u;n)]/u^n f(u;n)]| \leqq A n^l\,|u - 2\pi k|^{-l}.$$

Finally, equations (1.10) and (1.11) combine with Leibnitz's rule on the product $[u^n f(u;n)](\psi_n/\phi_n)(u)$ to obtain equation (1.10) for $m = s$.

Using (1.10) with $m = s$ in the same manner that (1.8) was used above, we readily obtain the theorem except for the interval $0 \leqq |u| \leqq \pi - \varepsilon_{n,s}$.

For the remaining case, assume that for $0 < |u| < \pi$ and $1 \leqq m < s$, there holds

$$(1.12) \qquad |D^m(\psi_n/\phi_n)(u)| \leqq A\,|n|^m\,|u^{n-m}|\,\pi^{-n}.$$

Since (1.11) holds even with $k = 0$, (1.12) and (1.11) combine with Leibnitz's rule to yield (1.12) for $m = s$. Then (1.12) can be used to finish the result. The theorem is proven.

COROLLARY 1.3 *For* $\pi - \varepsilon_{n,s} < |u| < \pi + \varepsilon_{n,s}$, $|(\psi_n/\phi_n)(u)| = \mathcal{O}(n^s)$.

## 2. The main results

I. J. Schoenberg [10] has shown a convergence result when the cardinal splines interpolate a function which is the Fourier-Stieltjes transform of a bounded measure on $[-\pi, \pi)$. (See Theorem 2.3 below.) The next step beyond measures is to obtain a theorem for Fourier transforms of distributions with support in $[-\pi, \pi]$, particularly, since the Paley-Wiener-Schwartz theorem (see [11, p. 311]) gives a 1-1 correspondence between entire functions of exponential type $\tau \leqq \pi$ with polynomial growth on the $x$-axis and Fourier transforms of such distributions. In a different context, R. Goldberg [1] essentially began the discussion that follows.

Let $y = \{y_\nu\} \in Y_s$, then there is a $2\pi$-periodic distribution $T^{\sim}$ whose Fourier series is

$$(2.1) \qquad T^{\sim} = \sum_{\nu=-\infty}^{+\infty} y_\nu e^{i\nu t}$$

where the series converges in the sense of tempered distributions. The Fourier transform of $T^{\sim}$ is given by

$$\mathscr{F}(T) = \sum_{\nu=-\infty}^{+\infty} y_\nu \delta_\nu \tag{2.2}$$

where $\delta_\nu$ is point evaluation at $\nu$, $\nu \in \mathbf{Z}$.

THEOREM 2.1. *Let $y = \{y_\nu\} \in Y_s$. Then the nth order cardinal spline, $\mathscr{L}_n y(x)$, which interpolates the data y at the integers is the Fourier transform of the tempered distribution $(T^{\sim} \psi_n/\phi_n)$ i.e.*

$$\mathscr{L}_n y = \mathscr{F}(T^{\sim} \psi_n/\phi_n). \tag{2.3}$$

PROOF. The cardinal spline is visibly given by the convolution $\mathscr{L}_n y(x) = (\mathscr{F}(T^{\sim}) * L_n)(x) = (\mathscr{F}(T^{\sim}) * \mathscr{F}(\psi_n/\phi_n))(x)$. Since $\psi_n/\phi_n$ is a quotient of cosine polynomials and $\phi_n(u) > 0$ (Schoenberg [6]), a standard theorem from distribution theory gives the result (see [11, chap. 30]).

We now determine how Theorem 2.1 can be used to help solve the convergence problem. The $2\pi$-periodic extension, $T^{\sim}$, of the distribution $T$ with support in $(-\pi, \pi)$ is essentially the (generalized) derivative of some $2\pi$-periodic function. The distribution theory will allow us to formally integrate by parts, and then apply the estimates in Theorem 1.1 on the derivates of $\psi_n/\phi_n$.

THEOREM 2.2 *Let $f(x)$ be the restriction to $\mathbf{R}$ of an entire function of exponential type $\tau < \pi$ which has polynomial growth on the real axis. Then, for any natural number s,*

$$\lim_{n \to \infty} |(\mathscr{L}_n f)^{(s)}(x) - f^{(s)}(x)| = 0 \tag{2.4}$$

*uniformly on bounded subsets of* $\mathbf{R}$.

PROOF. By the Paley-Wiener-Scwartz theorem, there is a unique distribution $T$ with support in $[-\tau, \tau]$ such that $\mathscr{F}(T)(x) = f(x)$. Further, there is a natural number $r$ and a continuous function $h$ with support in $(-\pi, \pi)$ such that for any test function $\phi$, $\langle T, \phi \rangle = \langle h^{(r)}, \phi \rangle = (-1)^r \langle h, \phi^{(r)} \rangle$. Let $T^{\sim}$ be the $2\pi$-periodic extension of $T$ and let $h^{\sim}$ be the corresponding $2\pi$-periodic extension of $h$. Let $D$ represent the differentiation operator.

Since $\psi_n/\phi_n$ and its derivatives to order $r$ decrease to zero as $|u| \to +\infty$ ($n > r + 2$ say), the function $\mathscr{L}_n f(x)$ may be written as

$$\mathscr{L}_n f(x) = \mathscr{F}(T^{\sim} \psi_n/\phi_n)(x) = \frac{(-1)^r}{2\pi} \int_{-\infty}^{+\infty} h^{\sim}(u) D^r [e^{ixu} \psi_n(u)/\phi_n(u)] du.$$

Also, $\mathscr{F}(T)(x) = [(-1)^r/2\pi] \int_{-\pi}^{+\pi} h(u) D^r [e^{ixu}] du$.

For $s=0$, we need to estimate

$$
\begin{aligned}
&|\mathcal{F}(T^{\sim}\psi_n/\phi_n)(x)-\mathcal{F}(T)(x)| \\
&\quad = (1/2\pi)\left|\int_{-\infty}^{+\infty} h^{\sim}(u)D^r[e^{ixu}\psi_n(u)/\phi_n(u)]du \right. \\
&\qquad \left. -\int_{-\pi}^{\pi} h(u)D^r[e^{ixu}]du\right|.
\end{aligned} \tag{2.5}
$$

Using Leibnitz's Rule for differentiation, and breaking the integration over intervals of the form $2\pi k-\pi \leqq u \leqq 2\pi k+\pi$, the various parts of (2.5) can be estimated. For $k \neq 0, \pm 1$, and $l \geqq 0$ an integer, a typical term of (2.5) will involve $D^l[(\psi_n/\phi_n)(u)]D^{r-l}[e^{ixu}]$, and Theorem 1.1 will allow the estimate

$$
\begin{aligned}
&|x|^{r-l}\int_{2\pi k-\pi}^{2\pi k+\pi} D^l[(\psi_n/\phi_n)(u)]\,|\,|h^{\sim}(u)|\,du \\
&\quad \leqq |x|^{r-l}n^l(3/2)^{-n}|k|^{-n}\int_{-\pi}^{\pi}|h(u)|\,du.
\end{aligned}
$$

Hence, the integration on $(-\infty,-3\pi]$ and $[3\pi,\infty)$ will be majorized by a finite linear combination of sums of the form $|x|^{r-l}n^l(3/2)^{-n}\sum_{|k|=2}^{+\infty}|k|^{-n}$, which tend to zero uniformly on bounded subsets of $\mathbf{R}$ as $n\to+\infty$.

On $[-3\pi,3\pi]$, the support of $h^{\sim}$ will not intersect intervals of length $2\pi(r+1)\log n/n$ centered on $\pm\pi$ for sufficiently large $n$. Thus, by equation (1.4) for $l>0$, $|D^l[(\psi_n/\phi_n)(u)]| = \mathcal{O}(n^{-1})$, and the terms in (2.5) involving these derivatives over the interval $[-3\pi,3\pi]$ will also converge to zero uniformly on bounded subsets of $\mathbf{R}$ as $n\to+\infty$.

The remaining two terms are dominated by

$$
(1/2\pi)|x|^r\left\{\int_{-\pi}^{\pi}|h(u)|\,|1-\psi_n/\phi_n(u)|\,du+\int_{\pi\leqq|u|\leqq3\pi}|h^{\sim}(u)|\,|\psi_n/\phi_n(u)|\,du\right\}
$$

which also converge to zero by nature of the support of $h$ [3, prop. 1.2]. Therefore, the theorem is proven for the case $s=0$.

The case $s>0$ introduces a power $t^s$ into the integrals to be estimated. On $[2\pi k-\pi, 2\pi k+\pi]$, the power $t^s$ is bounded by a constant times $|k|^s$. For large $n$, the $|k|^{-n}$ estimate in (1.3) dominates and the argument carries through as before.

In the case of a bounded measure, the estimates in proposition 1.2 of [3] and the fact that $\psi_n/\phi_n(\pm\pi)\to 1/2$ can be applied directly with Theorem 2.1 to yield the following theorem of Schoenberg without the restriction of $n$ even.

THEOREM 2.3. *Let* $f(x)=(1/2\pi)\int_{-\pi}^{\pi}e^{ixu}d\mu(u)$ *for* $\mu$ *a bounded measure on* $[-\pi,\pi)$. *If* $\mu$ *has no atom at* $-\pi$, *then* $\mathscr{L}_nf(x)$ *converges uniformly to* $f(x)$. *If* $\mu=\delta_{-\pi}$, *then* $\mathscr{L}_nf(x)$ *converges uniformly to* $\frac{1}{2}[f(x)+f(-x)]=\cos\pi x$.

## 3. An example

It is natural to ask whether the restriction to exponential type $<\pi$ can be dropped from Theorem 2.2. In any case, such convergence would have to be interpreted modulo an additive term of the form $P(x)\sin\pi x$. Recently, D. J. Newman [2] has provided a counter-example to uniform convergence for bounded exponential type $\pi$. Below we provide a counter-example to uniform convergence on bounded sets for exponential type $\pi$ and power growth on the axis.

For the data $z_\nu=(-1)^{\nu+1}$, $\nu\in\mathbf{Z}^+$, $z_\nu=(-1)^\nu$, $\nu\in\mathbf{Z}^-$ or $\nu=0$, Richards [4] has shown that $\mathscr{L}_nz(1/2)$ is asymptotic to $(2/\pi)\log n$. The formal Fourier series $\sum_{\nu=-\infty}^{+\infty}z_\nu e^{-i\nu t}$ exists as a tempered distribution and equals $1+g^{(2)}(t)$ where $g(t)$ is the continuous periodic function given by $g(t)=|\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{+\infty}[z_\nu/(i\nu)^2]e^{-i\nu t}$ ([12, corol. 2.4–3b, p. 51]). Let $T$ be the distribution defined by

$$\langle T,\phi\rangle=\int_{-\pi}^{\pi}\phi(t)dt+\int_{-\pi}^{\pi}g(t)\phi^{(2)}(t)dt.$$

The Fourier transform of $T$ is given by

$$\mathscr{F}(T)(x)=\frac{\sin\pi x}{\pi x}+\frac{(ix)^2}{2\pi}\int_{-\pi}^{\pi}g(t)e^{ixt}dt.$$

Thus, $\mathscr{F}(T)(\nu)=z_\nu$. This is an entire function of exponential type $\pi$ whose restriction to $\mathbf{R}$ is of power growth and which interpolates the data $z_\nu$ at the integers.

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